Higher Order Modification of the Schrödinger Equation

Waldemar Puszkarz*

Department of Physics and Astronomy,

University of South Carolina,

Columbia, SC 29208

(May 15, 1999)

Abstract

We modify the Schrödinger equation in a way that preserves its main properties but makes use of higher order derivative terms. Although the modification represents an analogy to the Doebner-Goldin modification, it can differ from it quite distinctively. A particular model of this modification including derivatives up to the fourth order is examined in greater detail. We observe that a special variant of this model partially retains the linear superposition principle for the wave packets of standard quantum mechanics remain solutions to it. It is a peculiarity of this variant that a periodic structure emerges naturally from its equations. As a result, a free particle, in addition to a plane wave solution, can possess band solutions. It is argued that this can give rise to well-focused particle trajectories. Owing to this peculiarity, when interpreted outside quantum theory, the equations of this modification could also be used to model pattern formation phenomena.

*Electronic address: puszkarz@cosm.sc.edu

Recently there has been a considerable interest in the Doebner-Goldin (DG) modification of the Schrödinger equation [1]. (See also [2] for a more complete list of references to this subject and [3, 4] for the latest update on the progress in a broader context related to the matter in question.) It is the purpose of the present letter to propose yet another modification of this fundamental equation that, similarly as the Doebner-Goldin modification, makes use of the current formulation and preserves the main features of the equation discussed, these of homogeneity [5] and weak separability of composed systems [6]. Moreover, a subset of equations of this modification, defined by certain values of its free parameters, complies with the Galilean invariance, giving rise to a special version of it that, except for linearity, possesses all the main properties of the Schrödinger equation deemed physically relevant.

Let us note that even though in some cases the homogeneity of nonlinear generalizations of the Schrödinger equation does entails their weak separability [7], the properties mentioned are, in general, independent [8]. As pointed out in [9], the homogeneity of nonlinear variants of this equation is essential for a unique defintion of their energy functionals. This property is also necessary for modifications of the Schrödinger equation to possess a weakly separable multi-particle extension [8, 10].

What differs our proposal from the DG modification is the use of terms that involve derivatives of the order higher than second and of higher polynomial degrees. As we will see, the simplest extension of this kind employes derivatives of fourth order and of second degree. The higher degrees are required for the completness of the formulation. As we will also see, a certain, physically most attractive, variant of this modification should be viewed as an extension of the Schrödinger equation rather than its modification for it changes the main properties of this equation less dramatically than the majority of the modifications. In particular, this true about its solutions.

To begin with, let us first reformulate the DG modification in a way that will be convenient for the intended generalization. To this end, let us write the modified Schrödinger equation as

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\Psi - \frac{i\hbar D}{2}F_{\{a\}}\left[\Psi,\Psi^*\right]\Psi + \hbar D F_{\{b\}}\left[\Psi,\Psi^*\right]\Psi, \tag{1}$$

where

$$F_{\{x\}}\left[\Psi,\Psi^*\right] = \sum_{i=1}^{n} x_i F_i\left[\Psi,\Psi^*\right]$$

and x_i are some dimensionless coefficients that form a generic array $\{x\}$ while $F_i\left[\Psi,\Psi^*\right]$ are functionals of Ψ and Ψ^* homogeneous of degree zero in these functions. The coupling constant D has the dimensions of the diffusion coefficient, meter^{2}second^{-1}, in the DG modification. In what follows, we will work with $\rho = R^2 = \Psi\Psi^*$ and S, the probability density and the phase of the wave function $\Psi = R\exp(iS)$, correspondingly. The general form of the functional employed by Doebner and Goldin is

$$F^{DG}_{\{x\}}\left[\rho, S\right] = x_1\Delta S + x_2\vec{\nabla}S\cdot\left(\frac{\vec{\nabla}\rho}{\rho}\right) + x_3\frac{\Delta\rho}{\rho} + x_4\left(\frac{\vec{\nabla}\rho}{\rho}\right)^2 + x_5\left(\vec{\nabla}S\right)^2. \tag{2}$$

The imaginary part of the Schrödinger equation leads to the continuity equation. The standard way to obtain it is to multiply both sides of the Schrödinger equation by Ψ^* and take the imaginary part of the ensuing expression. The result turns out to be

$$\frac{\partial\rho}{\partial t} + \frac{\hbar}{m}\vec{\nabla}\cdot\left(\rho\vec{\nabla}S\right) = 0 \tag{3}$$

with the probability current identified as $\vec{j} = \hbar\rho\vec{\nabla}S/m$.

In the same manner, we would like $\rho F_{\{a\}}$ to form the divergence of some current. One can show that two terms emerge to play this role: $\vec{\nabla} \cdot \left(\rho \vec{\nabla} S\right)$ and $\Delta\rho$. One obtains these in a unique way by putting $a_1 = a_2 = a$ and $a_4 = a_5 = 0$. Renaming $a_3 D \to D'$ and $aD \to D$ allows us to write the modified continuity equation as[1]

$$\frac{\partial \rho}{\partial t} + \frac{\hbar}{m}\vec{\nabla} \cdot \left(\rho \vec{\nabla} S\right) + D\vec{\nabla} \cdot \left(\rho \vec{\nabla} S\right) + D'\Delta\rho = 0 \tag{4}$$

which can also be put in the form

$$\frac{\partial \rho}{\partial t} + \frac{\hbar}{m^*}\vec{\nabla} \cdot \left(\rho \vec{\nabla} S\right) + D'\Delta\rho = 0, \tag{5}$$

where $m^* = m/\left(1 + Dm/\hbar\right) = m/\beta$ is the effective mass of a quantum system. Let us notice that the most uniform way to write the last equation is as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\hbar}{m^*}\vec{\nabla} \cdot \left(\rho \vec{\nabla} S\right) + D'\vec{\nabla} \cdot \left(\rho \frac{\vec{\nabla}\rho}{\rho}\right) = 0. \tag{6}$$

If we further note that $S = \frac{i}{2}\ln\left(\Psi^*/\Psi\right)$ and $\vec{\nabla}\rho/\rho = \ln \Psi^*\Psi$, we see that this way reveals the ubiquitous role of the logarithmic function in the DG modification[2] and suggests that the probability current be considered as consisting of two components having velocities $\vec{v}_S = \frac{\hbar}{m^*}\vec{\nabla} S$ and $\vec{v}_\rho = D'\vec{\nabla}\rho/\rho$. The latter is the diffusion component.

The linear Schrödinger equation is invariant under the Galilean transformation of coordinates $\vec{x} = \vec{x}' + \vec{v}t, t' = t$ provided the phase of the wave function transforms as

$$S(\vec{x}, t) = S'(\vec{x}', t') + m\vec{v} \cdot \vec{x}' + \frac{1}{2}m\vec{v}^2 t'. \tag{7}$$

One observes that the $D\vec{\nabla} \cdot \left(\rho \vec{\nabla} S\right)$ component of the additional current would break the Galilean invariance of the continuity equation if it were not for the mass redefinition. To ensure the Galilean invariance of this equation it is also required that the effective mass replaces the "bare" mass everywhere a reference to the latter is made, including in particular the phase of a quantum system in the above transformational formula. However, the redefinition in question by no means guarantees the Galilean invariance of the entire Schrödinger equation, which is maintained only if $D = 0$.

It is also easy to notice that a special version of DG modification can be straightforwardly derived by utilizing $\vec{\nabla} S$ and $\vec{\nabla}\rho/\rho$ as the components of a general vector potential $A_{DG} = d_1\vec{\nabla} S + d_2\vec{\nabla}\rho/\rho$. In the presence of A_{DG} the free Schrödinger equation reads

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - iA_{DG}\right)^2 \Psi. \tag{8}$$

One arrives from it at the linearizable variant of the DG proposal that generates the continuity equation in the form (4) or (5) and modifies the other part of the Schrödinger equation by adding terms $\left(\vec{\nabla} S\right)^2$,

[1] In the original DG formulation [1], D does not make its appearance at all. The case of nonzero D is treated in a subsequent generalization of their scheme involving nonlinear gauge transformations [3].

[2] This should not come as a surprise as the logarithmic function was already used for the sole purpose of ensuring that the modification of the Schrödinger equation proposed in [6] obeys the weak separability condition.

$\left(\vec{\nabla}\rho/\rho\right)^2$, and $\vec{\nabla}S\cdot\vec{\nabla}\rho/\rho$. The coefficients that stand by these terms depend only on two constants which are functions of d_i. It is clear now why this variant is linearizable and how to transform it into the linear Schrödinger equation. The latter is achieved by a nonlinear gauge transformation [3], similarly as this is done in electrodynamics. Some caution should be exercised when dealing with the $\vec{\nabla}S$ part of the gauge potential or the corresponding nonlinear gauge transformation. As pointed out above this part is responsible for the mass redefinition in the continuity equation. However, instead of the mass redefinition, one can also enforce the Galilean invariance of this equation by redefining the phase $S' = \beta S$. Now, we see that this alters the range of the phase from 2π to $2\pi\beta$ and should be taken into account when dealing with the uniqueness of the wave function Ψ.[3] Yet another way to ensure the Galilean invariance is to redefine the Planck constant, $\hbar \to \beta\hbar$, which eventually brings the nonlinear Schrödinger equation to the form of its linear prototype, except that with the rescaled Planck constant. This completes the linearization process. Being linearizable, the discussed variant of the DG modification is physically equivalent to the linear Schrödinger equation, and, in particular, it satisfies the Ehrenfest equations. The DG proposal in its full form is not linearizable. However, similarly as the variant in question is related to the linear Schrödinger equation via nonlinear gauge transformations and thus constitutes a kind of family with it, the equations of the full DG modification form a family that is closed under more general gauge transformations [3].

No stipulation is made on the coefficients $\{b\}$. However, if one wants the modified Schrödinger equation to be Galilean invariant, (7) implies the unique choice of $b_2 = b_5 = 0$. A restricted version of the DG modification for $b_2 = 0$ and $2b_3 - b_4 = 0$ in $F_{\{b\}}\left[\Psi,\Psi^*\right]$ was shown [8] to be derivable from a local Lagrangian density.

The main idea of our proposal is to extend the DG modification so as to include higher derivative terms. Despite an apparent simplicity of this task, the resulting construction turns out to be rather complex and can have diametrically different features than the proposal of Doebner and Goldin. We will first consider the leading order case, involving the derivatives up to the fourth order and the second degree, which, in addition to its simplicity, seems to be the most promising on physical grounds, and then suggest how to generalize this construction to allow also higher order terms. It is rather straightforward to convince oneself that

$$\begin{aligned}
F^{ext}_{\{x\}}\left[\rho,S\right] = {} & x_1\Delta\Delta S + x_2\Delta\left(\frac{\Delta\rho}{\rho}\right) + x_3\Delta\left(\frac{\vec{\nabla}\rho}{\rho}\right)^2 + x_4\Delta\left(\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}S\right) + \\
& x_5\Delta\left(\vec{\nabla}S\right)^2 + x_6\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}\left(\Delta S\right) + x_7\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}\left(\frac{\Delta\rho}{\rho}\right) + \\
& x_8\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\right)^2 + x_9\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}S\right) + x_{10}\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}\left(\vec{\nabla}S\right)^2 + \\
& x_{11}\vec{\nabla}S\cdot\vec{\nabla}\Delta S + x_{12}\vec{\nabla}S\cdot\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\right)^2 + x_{13}\vec{\nabla}S\cdot\vec{\nabla}\left(\frac{\Delta\rho}{\rho}\right) \qquad (9)
\end{aligned}$$

is the right choice for the homogeneous terms up to the fourth order. Terms $\left(\Delta S\right)^2$, $\Delta S\left(\Delta\rho/\rho\right)$, and $\Delta S\left(\vec{\nabla}\rho/\rho\right)^2$ have not been included. Even if homogeneous of degree zero in both Ψ and Ψ^*, they would nevertheless break the weak separability of the modified Schrödinger equation.

[3]See also [11] for a more profound discussion of this issue in the case the nonlinear gauge transformations are supposed to form a group.

To cast more light on this issue, let us demonstrate the weak separability of the Schrödinger equation in the hydrodynamic formulation. We are considering a quantum system made up of two noninteracting subsystems in the sense that [6]

$$V(\vec{x}_1, \vec{x}_2, t) = V_1(\vec{x}_1, t) + V_2(\vec{x}_2, t). \tag{10}$$

We will show that a solution of the Schrödinger equation for this system can be put in the form of the product of wave functions for individual subsystems for any $t > 0$, that is, $\Psi(x_1, x_2, t) = \Psi_1(x_1, t)\Psi_2(x_2, t) = R_1(x_1, t)R_2(x_2, t)exp\left\{i(S_1(x_1, t) + S_2(x_2, t))\right\}$ and that this form entails the separability of the subsystems. The essential element here is that the subsystems are initially uncorrelated which is expressed by the fact that the total wave function is the product of $\Psi_1(\vec{x}_1, t)$ and $\Psi_2(\vec{x}_2, t)$ at $t = 0$. What we will show then is that the subsystems remain uncorrelated during the evolution and that, at the same time, they also remain separated. It is the additive form of the total potential that guarantees that no interaction between the subsystems occurs, ensuring that they remain uncorrelated during the evolution. However, such an interaction may, in principle, occur in nonlinear modifications of the Schrödinger equation even if the form of the potential itself does not imply that. This is due to a coupling that a nonlinear term usually causes between $\Psi_1(\vec{x}_1, t)$ and $\Psi_2(\vec{x}_2, t)$. As a result, even in the absence of forces the very existence of one of the particles affects the evolution of the other one, clearly violating causality.

The discussed separability is called the weak separability since it assumes that the wave function of the total system is the product of the wave functions of its subsystems in contradistinction to the strong version of separability that does not employ this assumption. As shown by Lücke [12, 13], weakly separable modifications, such as the modification of Białynicki-Birula [6] or the Doebner-Goldin modification, [1] can still violate separability when the compound wave function is not factorizable, and thus they are not strongly separable. An alternative effective approach to the strong separability has been proposed by Czachor [14]. This approach treats the density matrix as the basic object subjected to the quantum equations of motion which are modified[4] compared to a nonlinear Schrödinger equation for the pure state. It admits a large class of nonlinear modifications including those ruled out by the fundamentalist approach advocated by Lücke and even those that are not weakly separable as, for instance, the cubic nonlinear Schrödinger equation.

The Schrödinger equation for the total system, assuming that the subsystems have the same mass m, reads now

$$\begin{aligned}
&\hbar\frac{\partial R_1^2 R_2^2}{\partial t} + \frac{\hbar^2}{m}\left\{\left(\vec{\nabla}_1 + \vec{\nabla}_2\right)\cdot\left[R_1^2 R_2^2\left(\vec{\nabla}_1 S_1 + \vec{\nabla}_2 S_2\right)\right]\right\} = \hbar R_2^2\frac{\partial R_1^2}{\partial t} + \hbar R_1^2\frac{\partial R_2^2}{\partial t} \\
&\quad + \frac{\hbar^2}{m}R_2^2\vec{\nabla}_1\cdot\left(R_1^2\vec{\nabla}_1 S_1\right) + \frac{\hbar^2}{m}R_1^2\vec{\nabla}_2\cdot\left(R_2^2\vec{\nabla}_2 S_2\right) = R_1^2 R_2^2\left\{\left[\hbar\frac{1}{R_1^2}\frac{\partial R_1^2}{\partial t} + \right.\right. \\
&\quad \left.\left.\frac{\hbar^2}{m}\frac{1}{R_1^2}\vec{\nabla}_1\cdot\left(R_1^2\vec{\nabla}_1 S_1\right)\right] + \left[\hbar\frac{1}{R_2^2}\frac{\partial R_2^2}{\partial t} + + \frac{\hbar^2}{m}\frac{1}{R_2^2}\vec{\nabla}_2\cdot\left(R_2^2\vec{\nabla}_2 S_2\right)\right]\right\} = 0
\end{aligned} \tag{11}$$

and

[4]What this means in practice is that the basic equation is the nonlinear von Neumann equation [15] instead of some nonlinear Schrödinger equation for the pure state.

$$
\begin{aligned}
&\frac{\hbar^2}{m}\left(\Delta_1+\Delta_2\right)R_1R_2-2\hbar R_1R_2\frac{\partial(S_1+S_2)}{\partial t}-\frac{\hbar^2}{m}R_1R_2\left(\vec{\nabla}_1S_1+\vec{\nabla}_2S_2\right)^2-\\
&\quad\left(V_1+V_2\right)R_1R_2=\frac{\hbar^2}{m}R_2\Delta_1R_1+\frac{\hbar^2}{m}R_1\Delta_2R_2-2\hbar R_1R_2\frac{\partial S_1}{\partial t}-2\hbar R_1R_2\frac{\partial S_2}{\partial t}\\
&\quad+\frac{\hbar^2}{m}R_1R_2\left(\vec{\nabla}_1S_1\right)^2+\frac{\hbar^2}{m}R_1R_2\left(\vec{\nabla}_2S_2\right)^2-V_1R_1R_2-V_2R_1R_2=\\
&\quad R_1R_2\left\{\left[\frac{\hbar^2}{m}\frac{\Delta_1R_1}{R_1}-2\hbar\frac{\partial S_1}{\partial t}+\frac{\hbar^2}{m}\left(\vec{\nabla}_1S_1\right)^2-V_1\right]+\left[\frac{\hbar^2}{m}\frac{\Delta_2R_2}{R_2}-2\hbar\frac{\partial S_2}{\partial t}+\right.\right.\\
&\quad\left.\left.\frac{\hbar^2}{m}\left(\vec{\nabla}_2S_2\right)^2-V_2\right]\right\}=0, \qquad (12)
\end{aligned}
$$

where we used the fact that $\vec{\nabla}_1\cdot\vec{\nabla}_2=0$. What we have obtained is a system of two equations, each consisting of terms (in square brackets) that pertain to only one of the subsystems. By dividing the first equation by $R_1^2R_2^2$ and the second one by R_1R_2, one completes the separation of the Schrödinger equation for the compound system into the equations for the subsystems. Moreover, we have also showed that indeed the product of wave functions of the subsystems evolves as the wave function of the total system. A similar analysis applied to (9) convinces us that the chosen functional does have the required property of weak separability. Of course, as already mentioned, this by no means guarantees that the separability will be maintained for nonfactorizable compound wave functions. However, the proposed modification should be strongly separable in the effective approach [14].

The coefficient D has now the dimensions meter^{4}second^{-1}. By choosing $a_1=a_6$, $a_2=a_7$, $a_3=a_8$, $a_4=a_9$, $a_5=a_{10}$, and $a_{11}=a_{12}=a_{13}=0$ one obtains the following continuity equation

$$
\begin{aligned}
&\frac{\partial\rho}{\partial t}+\frac{\hbar}{m}\vec{\nabla}\cdot\left(\rho\vec{\nabla}S\right)+D_1\vec{\nabla}\cdot\left(\rho\vec{\nabla}\Delta S\right)+D_2\vec{\nabla}\cdot\left[\rho\vec{\nabla}\left(\frac{\Delta\rho}{\rho}\right)\right]+\\
&D_3\vec{\nabla}\cdot\left[\rho\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\right)^2\right]+D_4\vec{\nabla}\cdot\left[\rho\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}S\right)\right]+D_5\vec{\nabla}\cdot\left[\rho\vec{\nabla}\left(\vec{\nabla}S\right)^2\right]=0, \qquad (13)
\end{aligned}
$$

where $D_i=a_iD$. The new currents are revealed to be

$$
\begin{aligned}
\vec{j}_1 &= D_1\rho\vec{\nabla}\Delta S,\ \vec{j}_2=D_2\rho\vec{\nabla}\left(\frac{\Delta\rho}{\rho}\right),\ \vec{j}_3=D_3\rho\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\right)^2,\ \vec{j}_4=D_4\rho\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}S\right),\\
\vec{j}_5 &= D_5\rho\vec{\nabla}\left(\vec{\nabla}S\right)^2. \qquad (14)
\end{aligned}
$$

The continuity equation (13) can also be written in the form

$$
\frac{\partial\rho}{\partial t}+\frac{\hbar}{m}\vec{\nabla}\cdot\left(\rho\vec{\nabla}S\right)+D\vec{\nabla}\cdot\left[\rho\vec{\nabla}F^{DG}_{\{a\}}\left[\rho,S\right]\right]=0. \qquad (15)
$$

Similarly as in the DG modification, no condition is put on the coefficients $\{b\}$, but if one requires the modification to be Galilean invariant one should demand that $b_4=b_5=b_9=b_{10}=b_{11}=b_{12}=b_{13}=0$. For the continuity equation to be Galilean invariant one needs to set $D_4=D_5=0$.

One can ask if using $\vec{\nabla} F_{\{a\}}^{DG}[\rho, S]$ as a vector potential in the linear Schrödinger equation would generate a special variant of the proposed modification similarly as it was demonstrated for the linearizable part of the DG equations. It turns out that this would not be so, which indicates that the modification discussed may not be linearizable and therefore may contain some new physics that cannot be described by the standard quantum theory. Such a formulation would contribute the square of the vector potential to the real part of the Schrödinger equation. However, none of its terms, as for instance $\left(\vec{\nabla}\Delta S\right)^2$, belongs to the modification under study, being either of the higher order or degree in derivatives. Therefore, to accomplish this, an extention of our proposal that would incorporate such terms into it would be necessary. Even though this seems to be rather a natural generalization of our work, it is not the purpose of the present paper, and because of that it will not be discussed here.

It is not clear if there exists a local Lagrangian for the modification proposed. It should be noted that not all equations of interest for mathematical physics are derivable from local Lagrangians, the best case in point being the celebrated Navier- Stokes equations with which no such Lagrangian can be associated [16]. It is also still unknown if the DG modification in its fully developed form can be derived from any local Lagrangian density.

An essential feature that differs this proposal from the DG modification is the existence of unmodified stationary states of the Schrödinger equation, characterized by $S = -Et/\hbar$, for a certain set of parameters. For this to occur, one needs to choose $D_2 = D_3 = 0$ and $b_2 = b_3 = b_7 = b_8 = 0$. If in addition to that we assume the Galilean invariance, the Hamiltonian of the nonlinear Schrödinger equation reflecting these constraints reads

$$H_{ME} = H_L + H_{NL} = H_L + \frac{i\hbar D_1}{2} \frac{\vec{\nabla} \cdot \left(\rho \vec{\nabla} \Delta S\right)}{\rho} + \hbar b_1 \Delta\Delta S + \hbar b_6 \frac{\vec{\nabla}\rho}{\rho} \cdot \vec{\nabla}\left(\Delta S\right), \tag{16}$$

where H_L denotes the Hamiltonian of the linear Schrödinger equation.[5] There may exist other stationary states as well. The energy of stationary states is defined as the expectation value of H_{ME}

$$E_{ME} = \int dV \Psi^* H_{ME} \Psi. \tag{17}$$

It is straightforward to show that this definition leads to

$$E_{ME} = E_L + \hbar\left(b_1 - b_6\right) \int dV \rho \Delta\Delta S, \tag{18}$$

where we assumed that the total current vanishes on a boundary in the infinity and where E_L is the expectation value of H_L given by

$$E_L = \int dV \left\{ \frac{\hbar^2}{2m} \left[\left(\vec{\nabla} R\right)^2 + \left(\vec{\nabla} S\right)^2 R^2 \right] + V R^2 \right\}. \tag{19}$$

The general condition for stationary states, $\partial\rho/\partial t = 0$, leads to the equation

$$\frac{\hbar}{m} \vec{\nabla} \cdot \left(\rho \vec{\nabla} S\right) + D_1 \vec{\nabla} \cdot \left(\rho \vec{\nabla} \Delta S\right) = D_1 \vec{\nabla} \cdot \left[\rho \left(\vec{\nabla}\Delta S + \omega \vec{\nabla} S\right)\right] = 0, \tag{20}$$

[5] The subscript ME stands for the **minimal** higher order **extension** of the Schrödinger equation as explained further.

where $\omega = \frac{\hbar}{D_1 m}$, which should be treated along with

$$\frac{\hbar^2}{m}\Delta\rho^{1/2} - 2\left(\hbar\frac{\partial S}{\partial t} + V + \frac{\hbar^2}{2m}\left(\vec{\nabla}S\right)^2 + \frac{\hbar b_1}{2}\Delta\Delta S\right)\rho^{1/2} - \hbar b_6\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}\left(\Delta S\right)\rho^{1/2} = 0. \tag{21}$$

Together, they constitute a system of equations from which new stationary states can be obtained. It is conceivable that these equations are satisfied by wave functions whose phase S depends also on the position. If $D_1 > 0$, one observes that even in the absence of the potential these equations imply the existence of a periodic structure with a period $2\pi/\sqrt{\omega}$ that emerges naturally as a harmonic solution for $\vec{\nabla}S$. It is a peculiarity of the discussed model that, unlike in some other physical situations, this structure is not assumed but appears as a consequence of the equations of motion. A free particle, in addition to a plane wave solution, would display solutions characteristic of a particle moving in a crystal, the band solutions. As observed in [17], an electron in a crystal is well focused in the same sense a particle in an accelerator is. The equation that governs strong focusing in accelerators is the same as the one that describes the motion of the electron in the periodic crystalline lattice and entails the band structure of solids. It is the Mathieu equation. In the discussed situation, one obtains this equation from (21) by putting $b_6 = 0$. The strong focusing that this equation implies may offer a novel solution to the problem of particle trajectory in quantum mechanics. As is well known, in standard quantum mechanics the particles are represented by wave packets. Yet, the wave packets in linear theory spread without limit, and in so doing they contradict the existence of sharp well defined particle trajectories that one observes in, say, bubble chambers. In nonlinear quantum mechanics this problem can, in principle, be solved. This is made possible by particle-like solutions, i.e., localized and non-spreading configurations known as solitons that are generic for nonlinear equations of motion. Being well localized, the solitons produce sharp trajectories in the bubble chamber. But, as we noted, the equations of the modification under study can possibly give rise to sharp well focused trajectories even if the particles are not solitons. Interestingly, it is the phase of the wave function that entails this remarkable solution, adding a new twist to the meaning of the wave-particle duality.

On the other hand, if treated outside the quantum theory, the discussed periodic solutions might provide a viable model for the pattern formation phenomena [18]. It should be noted in connection with this that the equations of the Doebner-Goldin modification have their predecessor in a phenomenological equation which originates in the study of the wave propagation in fluids and plasmas with sharp boundaries and dissipation [19].

As we see, with the assumptions of Galilean invariance and existence of unmodified stationary states, the number of undetermined constants of our modification reduces to only three, D_1 and b_1, and b_6. This particular variant of the modification supports ordinary Gaussian wave packets for which

$$S = \frac{mtx^2}{2\left(t^2 + t_0^2\right)} - \frac{1}{2}\arctan\frac{t_0}{t}, \tag{22}$$

(for simplicity in one dimension and in natural units) and the coherent states satisfying $\Delta S = 0$, and this again distinguishes it not only from the DG modification that does not allow for any of these packets, but also from the modifications of Staruszkiewicz [21] and the one proposed in [8] which admit only the coherent states, excluding the ordinary wave packets. The fact that the discussed variant admits ordinary wave packets indicates that its nonlinearity is weak. It is a general property of nonlinear modifications of the Schrödinger equation to exclude such packets. Since both ordinary Gaussian wave packets and coherent states constitute the result of superposing more elementary wave

functions, each of which is a solution to the modification concerned, this means that the variant in question maintains, if only partially, the linear superposition principle. We choose to call this novel and rare property among nonlinear modifications of the Schrödinger equation the weak nonlinearity. To the best of our knowledge, the property in question is shared by only two other nonlinear modification of this equation [21, 20]. We will call this variant of the modification the minimal higher order extension of the Schrödinger equation for it departs from this equation in the most minimal way, preserving all of its standard properties, including the stationary solutions.

One can easily show that the discussed version of the modification can also be put in the form that involves a vector potential $\vec{A} = a\vec{\nabla}\Delta S$ as follows

$$i\frac{\partial\Psi}{\partial t} = -\frac{\hbar}{2m}\left(\vec{\nabla} - i\vec{A}\right)^2\Psi - \frac{\hbar}{2m}\vec{A}^2\Psi + \frac{c_1\hbar}{2m}\vec{\nabla}\cdot\vec{A}\Psi + \frac{\hbar}{2m}\left[c_2 Re(\vec{A}\cdot\vec{\nabla}\Psi) + 2Im(\vec{A}\cdot\vec{\nabla}\Psi)\right]. \quad (23)$$

The constant $a = -\frac{mD_1}{\hbar}$ has dimensions meter^2, the other constants are defined as $c_1 = \frac{2mb_1}{a}$ and $c_2 = \frac{2mb_6}{a}$.

In general, the only notable exception to this rule being the modification of Białynicki-Birula and Mycielski, nonlinear modifications of the Schrödinger equation do not have the classical limit in the sense of the Ehrenfest theorem. The discussed variant of the modification is one of such cases. The nonlinear terms it introduces entail corrections to the Ehrenfest relations. We will now work out these corrections. For a general observable A one finds that

$$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\langle A\rangle_L + \frac{d}{dt}\langle A\rangle_{NL}, \quad (24)$$

where the nonlinear contribution is due to $H_{NL} = H_R + iH_I$, H_R and H_I representing the real and imaginary part of H_{NL}, respectively. The brackets <> denote the mean value of the quantity embraced. Specifying A for the position and momentum operators, one obtains the general form of the modified Ehrenfest relations [8]

$$m\frac{d}{dt}\langle\vec{r}\rangle = \langle\vec{p}\rangle + I_1, \quad (25)$$

$$\frac{d}{dt}\langle\vec{p}\rangle = -\left\langle\vec{\nabla}V\right\rangle + I_2, \quad (26)$$

where

$$I_1 = \frac{2m}{\hbar}\int dV\vec{r}\rho H_I, \quad (27)$$

$$I_2 = \int dV\rho\left(2H_I\vec{\nabla}S - \vec{\nabla}H_R\right). \quad (28)$$

In the derivation of the last formula it was assumed that $\int dV\vec{\nabla}\left(\rho H_I\right) = 0$ which indeed stems from the continuity equation. For the extension in question these integrals are found to be

$$I_{1ME} = D_1 m\int dV\vec{r}\vec{\nabla}\cdot\left(\rho\vec{\nabla}\Delta S\right), \quad (29)$$

$$I_{2ME} = \hbar\int dV\left[D_1\vec{\nabla}\cdot\left(\rho\vec{\nabla}\Delta S\right)\vec{\nabla}S - b_1\rho\vec{\nabla}\Delta^2 S - b_6\rho\vec{\nabla}\left(\frac{\vec{\nabla}\rho}{\rho}\cdot\vec{\nabla}\Delta S\right)\right]. \quad (30)$$

One can straightforwardly generalize this construction to even higher order derivatives. To this end, we note that the total current functional in (15) is what was the functional from which to build

the DG modification. Similarly, the next order current functional will be the functional of formula (9). The complexity of the higher order extensions makes them difficult to study. On the other hand, this may even not be physically justifiable or interesting. Equations involving fourth order derivatives do occur in nonlinear equations aimed at modeling physical phenomena, such as, for instance, the pattern formation [18]. Higher order derivatives are rather uncommon.

The question that deserves further study concerns the new currents. It is quite natural to inquire if they form any algebraic structure.Of particular interest is the issue of strong separability of the modification proposed in the fundamentalist approach.

It would also be interesting to compare this modification with the one put forward by Staruszkiewicz [21] and extended by this author [10] which is neither homogeneous nor possesses the property of weak separability and with the modification proposed in [8] which is homogeneous but does not automatically admit the weak separability of composed systems. The common feature of these modifications is the presence of higher degrees of derivatives or derivatives of the order higher than second. Similar comparisons would not be possible for modifications with lower degrees of derivatives. This provides a unique framework for a better understanding of the concepts of weak separability and homogeneity, their physical impact and possible connections.

The progress in these matters will be reported elsewhere.

Acknowledgments

I would like to thank Professor Paweł O. Mazur for bringing to my attention the work of Professor Staruszkiewicz which started my interest in nonlinear modifications of the Schrödinger equation and Professor Gerald Goldin for his interest in this work. A stimulating exchange of correspondence with Dr. Marek Czachor and a correspondence with Professor Wolfgang Lücke on the problem of separability in nonlinear quantum mechanics are is also gratefully acknowledged. This work was partially supported by the NSF grant No. 13020 F167 and the ONR grant R&T No. 3124141.

References

[1] H.-D. Doebner and G. A. Goldin, Phys. Lett. A **162**, 397 (1992); J. Phys. A **27**, 1771 (1994).

[2] P. Nattermann and R. Zhdanov, J. Phys. A **29**, 2869 (1996).

[3] H.-D. Doebner and G. A. Goldin, Phys. Rev. A **54**, 3764 (1996).

[4] H.-D. Doebner, G. A. Goldin, and P. Nattermann, J. Math. Phys. **40**, 49 (1999); preprint quant-ph/9709036.

[5] G. A. Goldin and G. Svetlichny, J. Math. Phys. **35**, 3322, (1995).

[6] I. Białynicki-Birula and J. Mycielski, Ann. Phys. (N. Y.) **100**, 62 (1976).

[7] S. Weinberg, Phys. Rev. Lett. **62**, 485 (1989).

[8] W. Puszkarz, *Nonlinear Phase Modification of the Schrödinger Equation*, preprint quant-ph/9710010.

[9] W. Puszkarz, *Energy Ambiguity in Nonlinear Quantum Mechanics*, preprint quant-ph/9802001.

[10] W. Puszkarz, *Extension of the Staruszkiewicz Modification of the Schrödinger Equation*, preprint quant-ph/9710009.

[11] M. Czachor, *On General Characterization of Nonlinear Gauge Transformations*, preprint quant-ph/9711053.

[12] W. Lücke, *Gisin Nonlocality of the Doebner-Goldin 2-Particle Equation*, preprint quant-ph/9710033.

[13] W. Lücke, *Nonlocality in Nonlinear Quantum Mechanics*, preprint quant-ph/9904016.

[14] M. Czachor, Phys. Rev. A. **57,** 4122 (1998).

[15] M. Czachor, M. Kuna, S. B. Leble, and J. Naundts, *Nonlinear von Neumann-Type Equations*, preprint quant-ph/9904110.

[16] B. A. Finlayson, Phys. Fluids. **13**, 963 (1972).

[17] T. R. Carver, Am. J. Phys. **39**, 1225 (1971).

[18] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[19] B. A. Malomed and L. Stenflo, J. Phys. A **24**, L1149 (1991).

[20] W. Puszkarz, *Relativistically Extended Modification of the Schrödinger Equation*, preprint quant-ph/9710008.

[21] A. Staruszkiewicz, Acta Phys. Pol. **B14**, 907 (1983).